Theoretical and experimental analysis of flow in a turbulent filter layer

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ABSTRACT: The paper is concerned with flow through a protective filter layer in order to predict the superficial velocity near a base layer. The analysis presented here is mostly analytical, relevant to the turbulent regime by approximation and based on length scales rather than diffusive theories. The salient result is that long wavelength fluctuations in the pressure in the open water problem over the filter are mostly responsible for variations in superficial velocity near the bottom of the filter. Various forms of experimental work confirm the theoretical analysis. Future work is outlined.

1 INTRODUCTION

Our research addresses the following (horizontal) situation. A base layer consisting of fine material is protected by a coarse-grained filter layer. The latter moderates a turbulent water flow applied to the top and the sides of the system. The criteria for initial motion of the base particles obviously depends on the flow conditions close to the base layer. In order to design the filter layer (its thickness and its granular content) it is necessary to ascertain the characteristics of the flow profile through the coarse material and in particular how the design impacts on the flow in the vicinity of the base/filter interface. A one-dimensional description has been put forward in the past by Shimizu et al (1990), treating a vertical cross-section through the filter layer and attempting to model the diffusive behaviour of fluctuations in the turbulent porous flow. While from a practical point of view this approach would appear to be very attractive as only velocity and fluctuations need to be specified at the top of the medium, the physical basis for such a theory is very doubtful, for the experimentally observed phenomena are all consistent with higher-dimensional effects. Furthermore, the stress equilibrium argument deployed by Shimizu et al (1990) ignores the skeletal stress of the packed filter bed, which, again, is physically unreasonable.

As an alternative model a more traditional approach is taken, using the well-founded equation of continuity and the Forcheimer relation, relating the pore pressure gradient $\nabla p$ to the filter velocity $v$ (that is, the time-averaged, multi-pore averaged, superficial velocity)

$$\frac{\partial p}{\partial x_i} = -C_i v_i - C_t v |v| v_i$$

(1)

Here the permeabilities $C_i$ and $C_t$ are both positive. The subscripts under the coefficients $C$ are indicative of the flow regime: $\ell$ denotes laminar and $t$ turbulent. Using Einstein’s summation convention, the equation of continuity reads:

$$\frac{\partial v_i}{\partial x_i} = 0$$

(2)

To make contact with the stress analysis by Shimizu et al (1990) it is noted in passing that the mean shear stress exerted on the walls of the pores is proportional to $v$.

Equations (2) and (3) form a complete description, in that there are as many equations as unknowns. Key question is: what are the boundary conditions?

2 BOUNDARY CONDITIONS

The boundary conditions are discussed in the context of the following physical principles

1. Across any boundary the sum of forces must be zero.
2. Across any boundary the equation of continuity must hold.
When these conditions are applied to any boundary that separates one densely packed granular medium (1) from another one (2), the usual groundwater flow formulas emerge. These are

\[ v_{\perp}^{(1)} = v_{\perp}^{(2)} \quad \text{and} \quad p^{(1)} = p^{(2)} \]  

The conditions for transitions from open water to a granular medium are less simple, involving an assessment of the flow properties of the open water flow system. To begin with the flow continuity is treated. If the granular medium (1) has porosity \( n^{(1)} \), and recalling that the porous medium equations deal with multi-pore averages, then flow continuity requires that

\[ v_{\perp}^{(1)} n^{(1)} = \langle v_{\perp} \rangle^{(2)} \]  

where the averaging brackets in the open water are such that a good number of nearby pores are covered. This boundary condition is especially useful when a boundary is made completely impermeable by artificial means.

When there is a transition between the granular medium and open water flowing over it, a pressure-based boundary condition must be contemplated. These may be associated with two causes. The first is the pressure variation that follows the large scale variation in flow in the neighbourhood of the filter. The second is the roughness of the boundary which causes surface pressure effects. The latter pressure effects are present due to turbulent flow dissipation that is out of phase with the orographical variations in the terrain and cause a ‘pressure drag’ on the flow. These phenomena have been described extensively in the meteorological literature, see for example Wood & Mason (1993). Generally speaking however, the Reynolds numbers in meteorological applications conditions are higher than the ones in grain-size hydraulic applications (for coarse gravel by about a factor of 10-100) and as a result the magnitude of the spatial variation is negligible compared to the large scale pressure variations.

Effects that are manifest on a scale that is large compared to the gravel grain size are important. Naturally these do not take place for unhindered flow parallel to the open water surface: there must be a physical reason for the pressure variations. For example when the water flow is forced past an obstacle. It is noted that such a geometry immediately introduces mixed boundary conditions, as the obstacle will be impermeable.

The solution of the pressure distribution for a given open water flow presents a problem. Traditional approximations to the Navier Stokes equation in the turbulent regime tend to be phrased such that the pressure variations are not part of the description. The popular \( k-\varepsilon \) method, for example, requires substantial adaptation to deliver a pressure field. While the practical Reynolds numbers are not terribly high (<10,000), so that a full-scale implementation of the Navier-Stokes equations is possible, a computational way-out can be found. For higher numbers physical scale-model testing will have to be done. This is of course a well-known problem in fluid mechanics.

As an example of a Navier Stokes solution the result of a lattice-Boltzmann calculation is shown here at low Reynolds number, see Figure 1.
For these low Reynolds numbers temporal fluctuations are small, but for higher ones they become manifest (the lattice Boltzmann method can be pushed to a Reynolds number of some 100,000 in two dimensions with ‘common’ computer power). The pressure at the gravel/open water interface is plotted in Figure 2. Fundamentally the result is that the pressure ‘jumps’ across the obstacle, causing a spatial groundwater flow velocity that involves variations on a scale of the order of the size of the obstacle. As a result there will be substantial superficial velocity variations in the filter layer; these extend to a depth of the order of magnitude of the size of the obstacle.

3 SOLUTIONS FOR THE BASIC EQUATIONS, SAMPLE PROBLEM

Equations (1) and (2) are now solved approximately for the following geometry: a rectangular area of filter material on top of a fine sand layer; \(-L < x < L; 0 < y < H\), Figure 3. The approximation is that the speed \( |v| \) is roughly constant in the filter layer. Equation (1) then reads

\[
\frac{\partial p}{\partial x_i} = \left(C_i v_i + C_i |v| \right) v_i \equiv -C v_i
\]

(5)

The problem is now linear. Improvements on this approximation will be discussed below.

The length scales of the problem are clearly the following (see Figure 3, bottom)

1. The pore size/grain size. This scale in not accessible to the current analysis.
2. A depth scale \( H \); this is a design parameter. Note that the analysis here will only deal with a filter thickness that is large compared to the grain size (say five times).
3. A length scale \( L \). This size is given by the dimension of the structure.

\[\begin{array}{c}
\text{Figure 3. Top: geometry of the sample problem.} \\
\text{Bottom: length scales of the problem.}
\end{array}\]

4. A length scale, or a set of length scales, associated with the variation in the pressure conditions at the top of the filter layer. This length scale will be called \( \mu \).

The interplay of these scales is obviously important as it concerns the design of the thickness of the protective filter. The linearisation introduced in Equation (5) will not always work in the sense that the average speed may vary substantially throughout the medium. The analysis given here may then only have first order status.

A strategy to deal with the non-linear character of the problem will be given below, which should at least indicate when a more involved analysis is required.

Introduce a stream function in the usual manner:

\[
v_x = -\frac{\partial \psi}{\partial y}; \quad v_y = \frac{\partial \psi}{\partial x}
\]

(6)

Now,

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0
\]

(7)

The boundary condition at the base/filter is easily implemented. The base material is assumed to be much less permeable than the filter layer and the first of Equation (3) approaches to the simple value \( \psi(0,0) = 0 \), the solution of (6) is then:

\[
\psi(x,y) = \int_0^\infty A(\lambda) \cos \left(\frac{2\pi \lambda}{\lambda}\right) + B(\lambda) \sin \left(\frac{2\pi \lambda}{\lambda}\right) \sinh \left(\frac{2\pi \lambda}{\lambda}\right) d\lambda
\]

(8)

Here \( A(\lambda) \) and \( B(\lambda) \) are functions of a wavelength parameter \( \lambda \). These functions determine the outcome of the problem; in particular it is important to know what the velocity at the bottom of the filter layer is. This quantity is obtained from (5) with the solution (7), to give

\[
v_x(x,0) = -2\pi \int_0^\infty A(\lambda) \cos \left(\frac{2\pi \lambda}{\lambda}\right) + B(\lambda) \sin \left(\frac{2\pi \lambda}{\lambda}\right) \lambda^{-1} d\lambda
\]

(9)

In order to acquire some insight in the structure of the function \( A(\lambda) \) and \( B(\lambda) \) various physical quantities at the boundaries of the problem are
determined. The velocity normal to the surface at the top of the filter layer is

$$v_y(x, H) = 2\pi \int_0^\infty d\lambda \lambda^{-1} \sinh \left( \frac{2\pi H}{\lambda} \right) \times$$

$$\times \left[ -A(\lambda) \sin \left( \frac{2\pi x}{\lambda} \right) + B(\lambda) \cos \left( \frac{2\pi x}{\lambda} \right) \right]$$

(10)

The pressure at the top of the filter layer is

$$p(x, H) = p(0, 0) + C \int_0^\infty \left[ A(\lambda) \sin \left( \frac{2\pi x}{\lambda} \right) - B(\lambda) \cos \left( \frac{2\pi x}{\lambda} \right) \right] \cosh \left( \frac{2\pi H}{\lambda} \right) d\lambda$$

(11)

For future use the average pressure gradient as a function of the height $y$ is determined

$$\left\langle \frac{\partial p}{\partial x} \right\rangle_y = \frac{1}{2L} \int_0^L \frac{\partial p}{\partial x}(x, y) dx = \frac{C}{L} \int_0^\infty A(\lambda) \sin \left( \frac{2\pi L}{\lambda} \right) \cosh \left( \frac{2\pi y}{\lambda} \right) d\lambda$$

(12)

As well as the pressure as a function of the depth at the point $x = 0$

$$p(0, y) = p(0, 0) + C \int_0^\infty \left[ B(\lambda) \cosh \left( \frac{2\pi y}{\lambda} \right) - B(\lambda) \right] d\lambda$$

(13)

This function can be measured using pore pressure gauges that are buried at a number of locations in the gravel of the filter layer.

A simple effect is immediately read from Equation (13): if $B(\lambda)$ is non-zero for large wavelengths only, such that $\lambda/2\pi \gg H$, then the integral will not contribute for any value of $y$ and hence $p(0, y)$ is a constant. In other words, if a section at the top of the filter, with a size greater than the depth of the filter, is made impermeable by artificial means, then the pore pressure will be more or less constant as a function of depth. This fact represents a simple experimental test on the correctness of the analysis put forward here, see further below in the section EXPERIMENTAL.

An important case is the behaviour of flow in the wake of an obstacle. Figure 2 provides inspiration for a sample problem. The pressure distribution at the top of the filter layer is here roughly of the form

$$p(x, H) = p_0 e^{-x/\mu}$$

(14)

where $\mu$ is a measure for the size of the obstacle. It is easily derived from the pressure boundary conditions at the top of the filter layer that the functions $A(\lambda)$ and $B(\lambda)$ have the following form, expressed in $\xi$, rather than $\lambda$, with $\xi = 2\pi/\lambda$

$$A(\xi) = \text{ctst} \frac{2\xi}{\xi^2 + \mu^{-2} \cosh(H\xi)}$$

(15)

$$B(\xi) = -\text{ctst} \frac{2\mu^{-1}}{\xi^2 + \mu^{-2} \cosh(H\xi)}$$

(16)

At the same time integrals over $\lambda$ become integrals over $\xi$, without penalty to the formalism.

Figure 4 (top) Resulting superficial velocity at the edge of the base layer for three ratios of the filter thickness to pressure fluctuation length scale.

Figure 5 (bottom) Transfer function: the ratio of the superficial velocity at the edge of the base layer to the magnitude of the pressure fluctuation at the top of the filter layer.
The all-important function \( v_t(x,0) \) is evaluated by numerical means and depicted in Figure 4. The values for the various parameters have been chosen such that the length scale of the pressure variation at the top is made unity: \( \mu = 1 \); the ‘other’ length scale, \( H \), can now be varied by way of parameter sensitivity.

It is observed that the variation in velocity depends quite strongly on the thickness of the filter layer. In fact, for filter thicknesses that are much smaller than the length scale of the pressure fluctuations a huge velocity fluctuation is present.

Such behaviour begs the question if it is possible to write down some simple measure for the transfer of the fluctuations from the pressure variations at the top to velocity fluctuations at the bottom. Inspecting the formulae it is seen that such a transfer function must have the form

\[
\left| \frac{\hat{v}_x(0)}{\hat{p}(H)} \right| C H = \frac{H}{\lambda \cosh(2\pi H/\lambda)}
\]  

(17)

(amplitudes are here denoted by a hat).

A plot of the right hand side of this equation is provided as a function of \( \lambda / H \). This plot shows very clearly that long wavelength fluctuations are transmitted much more vigorously than short wavelength ones. The wavelength is measured in terms of the thickness of the filter layer.

The conclusion of all this is that design should be carried out on the basis of the expectation of the largest length scale in the spatial variation of the pressure variations at the top of the filter.

One more comment can be made à propos of the simple linear analysis. A transfer function relating the vertical velocity variations at the top of the filter and the horizontal velocity variation at the bottom of the filter. This is essentially what Shimizu et al (1990) attempted to achieve. While it must be stressed that these authors did not in any way appreciate the higher dimensional character of the problem they arrive at an equation which has the following form with a set of positive coefficients \( \alpha \).

\[
\alpha_0 \frac{\partial^2 v_x}{\partial y^2} - \alpha_1 v_x - \alpha_2 v_x^2 = 0
\]  

(18)

To solve this equation boundary conditions need to be specified and at this point the treatment by the authors falls down, in that they are prepared to specify one boundary condition at the top of the filter, but none at the bottom. To an extent the problem can be circumvented by specifying the average velocity in the medium. None of this is meant to ameliorate the problems there are in this theory, but the authors claim that their theory broadly speaking agrees with experiment. An explanation for this is found in the fact that the distribution of signs in Equation (18) is carefully selected so that a transfer function appears to coincide with the correct form. The correct form must be the one that follows from the above formulas:

\[
\left| \frac{\hat{v}_x(0)}{\hat{v}_y(H)} \right| = \frac{1}{\sinh\left(\frac{2\pi H}{\lambda}\right)}
\]  

(19)

Linearisation of Equation (18) gives a similar possible transfer function (boundary conditions are not strictly speaking required for the evaluation of a transfer function).

4 A FURTHER INVESTIGATION OF THE NONLINEARITY

The nonlinear character of the equations for the groundwater flow is due to the turbulent term. Its approximate solution the previous paragraph relies on the assumption that the speed \( v \) is approximately constant throughout the filter. A further refinement takes account of variation of the speed in the medium. At this stage the intention is merely to investigate the effect of the nonlinearity and therefore a somewhat simplified approach is taken in which the laminar term in the constitutive equations is neglected.

The equation of continuity (2) does not change upon the introduction of the nonlinear term

\[
\frac{\partial v_i}{\partial x_i} = 0
\]  

(2)

Therefore the stream function \( \psi \) still exists. The constitutive equation, Equation (1) is now expressed in terms of the stream function:

\[
\frac{\partial p}{\partial x} = C_i |\psi| \frac{\partial \psi}{\partial y}
\]  

(20)

\[
\frac{\partial p}{\partial y} = -C_i |\psi| \frac{\partial \psi}{\partial x}
\]  

(21)

Eliminate \( p \) from these equations to leave:

\[
\frac{\partial |\psi| \frac{\partial \psi}{\partial y}}{\partial y} + \frac{\partial |\psi| \frac{\partial \psi}{\partial x}}{\partial x} + |\psi| \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = 0
\]  

(22)

Dividing Equation (22) by the speed yields the following equation
\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial \psi}{\partial y} \frac{\partial \ln(\frac{|\psi|}{|\psi_0|})}{\partial y} - \frac{\partial \psi}{\partial x} \frac{\partial \ln(\frac{|\psi|}{|\psi_0|})}{\partial x} \quad (23)
\]

where \(|\psi_0|\) is a reference speed (for example the average speed in the problem). This may be further approximated to adapt to a practical situation. For example, assume that a key term is \(-\frac{\partial \psi}{\partial y} \frac{\partial \ln(|\psi|)}{\partial y}\)

and let \(|\psi| = |\psi_0| + \beta y\) with \(\beta\) not too wild; then

\[-\frac{\partial \psi}{\partial y} \frac{\partial \ln(|\psi|)}{\partial y} \rightarrow -\frac{\partial \psi}{\partial y} \frac{\beta}{|\psi_0|}.\]

The equation to be solved is then:

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial \psi}{\partial y} \frac{\beta}{|\psi_0|} \quad (24)
\]

Now suppose that a length scale is present in the problem of wavelength \(\lambda\). Expressing all length scales in terms of \(\lambda\) now results in the scaled equation

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial \psi}{\partial y} \left( \frac{\beta}{|\psi_0|} \lambda \right) \quad (25)
\]

It is immediately seen that the nonlinearity becomes manifest when the nondimensional parameter

\[
T = \left( \frac{\beta}{|\psi_0|} \lambda \right) \quad (26)
\]

is greater than 1. (For a problem in two dimensions there will be two coefficients \(T\))

This will happen when the speed gradient in the problem is large and when the wavelength of the fluctuations in the problem are large. These two tend to balance one another as the gradient will be of the order of a fraction of the average speed to the largest length scale, making \(T\) a small number.

As a result the linear analysis of the previous Section is a reasonable approximation to the whole problem and, moreover, gives very reasonable insight in the qualitative properties of the problem. An issue that is not addressed here is that it is in principle possible that the problem has a mix of length scales, which may lead to nonlinear coupling. For these problems a numerical solution of equations (1) and (2) must be found.

5 EXPERIMENTAL

The experimental evidence for the theoretical principles set out in the previous sections come from two sources. The first represents experimental results by Shimizu et al (1990). The measured transfer function indicates a length scale of the variation of the external flow of the order of 1-2 cm. This is also the length scale of the gravel used in the experiments and the apparatus is carefully constructed to avoid any other length scales playing a part. This experiment – though widely quoted – does not give a lot of insight in any real-life constructions.

A second source is as yet unpublished material by Verheij (1997). These experiments do have an obstacle in them and are further enhanced by a series with an impermeable plate, the dimensions of which are much larger than the grain size of the gravel. The variation in the pore pressure that is displayed in these modestly termed ‘orientating experiments’ is in very good agreement with the theoretical effects presented in this paper.

A further set of experiments is being conducted at the moment. These are much more involved than any previous ones. Fundamentally a modification of the apparatus developed at Karlsruhe BAW is employed, see Köhler et al (1996). Pore pressures, both inside and outside the gravel layer are measured. Pore fluid velocities are estimated by endoscopic means. In this way a unique picture is constructed, enabling the determination of constants in the theory. These experiments are then intended to facilitate the step from qualitative to quantitative predictions and thus to accurate design of the thickness of the filter layer given the external flow conditions.

6 CONCLUSIONS

In the past much work has been devoted to the measurement of criteria for initial motion near base layers. These experiments have always been on the basis of homogeneous flow conditions. Well-known design formulae such as Bakker’s formula take account of the grain sizes involved as well as the hydraulic conditions, see Bakker et al (1994). This is a great improvement over mere geometric design, which ignores any safety from insufficient strength of the hydraulic force.

The work presented here is motivated by the idea that the same, or similar, formulae can be used for nonhomogeneous flow conditions, by ascertaining the local properties of the flow given external flow data. The analysis presented here is based on length scales of fluctuations as these come to the fore in the simplest nontrivial theory. While further experimental verification is on the way, the basic parametric sensitivities of the transmission of
external pressure fluctuations to the surface of the base material have been charted.

Future work will include endoscopic inspection, not only of the flow field in the pores of the filter, but also of the filter base layer interface in order to ascertain initial, or possibly and continual, base particle motion.

Another area, not touched on this paper, is the question of explicit time dependence. The turbulent flow in the open water will display time-dependent features that have non-vanishing multi-pore amplitudes. The amplitude will doubtlessly be attenuated in the filter, but an estimate of the extent to which this will occur needs further investigation, before it is adequately incorporated into the safety assessment of the design.

7 REFERENCES


